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Observables of asymptotically vanishing correlations, states at infinity and quantum separability

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Abstract. The asymptotic behaviour of quantum mechanical states at large times has been discussed in two recent papers by Wan and McLean. In a third paper a detailed study of the corresponding behaviour of observables was made. The mathematical properties of asymptotic operators obtained are applied in this paper to establish an algebraic formulation of quantum mechanics which has the characteristic of being asymptotically separable. The age old non-locality problem in quantum mechanics can then be effectively tackled.

1. Introduction

The algebraic approach to quantum field theory and to infinite quantum systems is well known and well established (Haag and Kastler 1964, Emch 1972, Bogolubov *et al* 1975, Bratteli and Robinson 1979, 1981). The idea is to associate a C^*-algebra with a physical system so that the self-adjoint elements of the algebra correspond to observables and positive linear functionals on the algebra correspond to states of the system. Such an approach is also applicable to quantum mechanical systems, i.e. systems consisting of a fixed and finite number of particles (Segal 1947, Bogolubov *et al* 1975). The simplest scheme which can reproduce the principles of quantum mechanics in Hilbert space is to identify the C^*-algebra with the von Neumann algebra $B(\mathscr{H})$ of all bounded operators on the Hilbert space $\mathscr{H}$ associated with the system. Observables are then the self-adjoint members of $B(\mathscr{H})$ and states are describable by normalised positive linear functionals on $B(\mathscr{H})$.

What we shall present in this paper is a theory of quantum mechanical systems formulated in terms of the algebraic methods in which the C^*-algebra is identified with a proper subset of the set $B(\mathscr{H})$ of all bounded operators in the Hilbert space $\mathscr{H}$. The theory leads to the notion of states at infinity and to quantum separability at infinity, hence a resolution of the de Broglie paradox in quantum mechanics. The physical ideas of the theory will be set out as a list of assumptions in the next section while the precise mathematical formulation will be given in § 3. We should point out that our theory is formulated in a non-relativistic form with space and time clearly separated. Any localisation referred to means a localisation in space.

2. Physical considerations

A measuring device is invariably of finite spatial size and a measurement is always performed within a finite time interval. It is then reasonable to assume that a measurement process should not be able to bring two states which are infinitely separated (in space) to interfere with each other. We shall formalise this by making the following assumption.

Assumption 1. No measurement process can bring two infinitely separated states to interfere with each other.

Now a measurement process is an operation to ascertain the values of certain observables. To be consistent with assumption 1 we shall exclude those observables in the conventional quantum mechanics which can correlate two infinitely separated states. Again we shall formalise this by the following assumption.

Assumption 2. No observable can correlate two infinitely separated states.

Obviously the assumptions stated above are intuitive ones; their meaning will be made precise as we go along.

Let us consider firstly the idea of infinitely separated states. Such an idea is necessarily an asymptotic concept. In two recent papers Wan and McLean (1983a, b, hereafter referred to as papers I and II) introduce a precise definition of asymptotic localisation and of asymptotic separation of states in quantum mechanics. Asymptotic operators were also introduced recently by Wan and McLean (1984, hereafter referred to as paper III). We shall adhere to the notation used in our papers I, II and III. We can now make use of the concepts introduced in these papers to enlighten our intuitions in assumptions 1 and 2.

Using the notion of asymptotically separable states (papers I, II) we can visualise two infinitely separated states as two asymptotically separable states ϕ_t, ϕ'_t in the limit as t tends to infinity. Using the notion of asymptotic observables we can visualise an observable admissible by assumption 2 to be firstly an asymptotic operator to enable us to talk about the expectation value of the observable as $t \to \infty$, and to be secondly an operator of asymptotically vanishing correlations to give a vanishing correlation $\langle \phi_t | A\phi'_t \rangle$ of two asymptotically separable states ϕ and ϕ' as t tends to infinity (paper III).

We have now stated, albeit in imprecise terms, the necessary physical requirements. Quantum mechanics as conventionally formulated in Hilbert space violates these requirements. In § 3 we shall present a theory to meet these physical requirements.

3. Theory

We shall now present an algebraic formulation by associating with a quantum mechanical system a C^*-algebra which is a C^*-subalgebra of $B(\mathscr{H})$. The reason for taking only a subalgebra of $B(\mathscr{H})$ is because we want to exclude operators which do not satisfy our assumption 2. For simplicity we shall present here a theory applicable for a free quantum particle with configuration space $\mathbb{R}^n$.

3.1. Postulates

The formal statements of our formulation are set out in the first instance in the following two postulates.

Postulate 3. Any state of the system is represented by a convex combination $\lambda\omega_\rho +$ $\mathscr{A} = \mathscr{A}^s_{\text{avc}} \cap \mathscr{A}^s$ of operators on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$ (see paper III for notation). Bounded observables correspond to self-adjoint members of $\mathscr{A}$.

Postulate 2. A state of the system is represented by a normalised positive linear functional ω (NPLF for short) on $\mathscr{A}$. If $A \in \mathscr{A}$ is self-adjoint then $\omega(A)$ is the average (expectation) value obtained from measurement of the observable represented by A in the state represented by ω.

Note that $\mathscr{A} = \mathscr{A}^s_0 + L^\infty(\boldsymbol{p})$ (paper III) and that an NPLF on $\mathscr{A}$ is a map ω: $\mathscr{A} \to \mathbb{C}$ satisfying (1) $\omega(I) = 1$, I being the identity operator, (2) $\omega(A^\dagger A) \geq 0\ \forall A \in \mathscr{A}$ and (3) $\omega(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1\omega(A_1) + \lambda_2\omega(A_2)\ \forall A_1, A_2 \in \mathscr{A}$ and $\forall \lambda_1, \lambda_2 \in \mathbb{C}$ (Haag 1972).

3.2. On the observables

Many of the mathematical properties of $\mathscr{A}$ are given in paper III. We have chosen $\mathscr{A}$ in setting up postulate 2 because it is the maximal algebra among the weak, the strong and the uniform topologies on $B(\mathscr{H})$ possessing the three desired properties, i.e. $\mathscr{A}$ is a C^*-algebra as well as asymptotic algebra and each element of $\mathscr{A}$ has asymptotically vanishing correlations.

For an idea of the kind of operators included in $\mathscr{A}$ we would mention quasi-local operators in $\bar{\mathscr{A}}_L$ (paper III), compact operators and all finite-dimensional projectors in particular. An example of familiar observables excluded because of the requirement that an observable must belong to $\mathscr{A}^s_{\text{avc}}$ is the parity operator $\mathscr{P}$ defined on $\mathscr{H}$ by $\mathscr{P}\phi(\boldsymbol{x}) = = \phi(-\boldsymbol{x})$. $\mathscr{P}$ commutes with the free particle Hamiltonian H_0, but not with the momentum $\boldsymbol{p}$. We can intuitively see that $\mathscr{P}$ can correlate states large distances apart. Anyone unduly alarmed by the exclusion of the well known parity operator may be reminded that we do still have local parity operators. For a discussion of local observables we refer to Wan and Jackson (1983) and Wan *et al* (1983). The whole point of excluding observables with infinite spatial correlations is to enable us to construct a theory which is asymptotically separable while retaining locally the essential features of the existing quantum mechanics. All this will become clear as we go on.

Finally as an example of an (unbounded) observable (an unbounded observable may be introduced as a self-adjoint operator on $\mathscr{H}$ whose spectral projectors are all in $\mathscr{A}$) excluded because of the asymptotic convergence requirement we mention the position operator $\boldsymbol{x}_t = (\boldsymbol{p}t/m) + \boldsymbol{x}$ which obviously does not converge as t tends to infinity.

3.3. On states

Our postulate 2 conforms with the usual practice in algebraic theories. However it may well be that not every NPLF should give rise to a state as this may create many unphysical states (Primas and Müller-Herold 1978). In order to bring out the physical ideas and results quickly and simply we shall in this paper restrict the states to a subset of NPLFS consisting of all the normal NPLFS and the normal NPLFS at infinity defined below.

Definition 1. Normal states. An NPLF ω on $\mathscr{A}$ is normal if ω corresponds to a density operator ρ on $\mathscr{H}$ in that $\omega(A) = \text{Tr}(\rho A)$ for every $A \in \mathscr{A}$.

Obviously every one-dimensional projector P on $\mathscr{H}$ defines a normal NPLF by $\omega(A) = \text{Tr}(PA)$. $\text{Tr}(PA)$ is of course just the expectation value $\langle\phi|A\phi\rangle$ where ϕ is a normalised vector in $\mathscr{H}$ associated with the projector P. We shall denote this NPLF by ω_ϕ. More generally we shall write ω_ρ.

Definition 2. An NPLF ω^∞ on $\mathscr{A}$ will be called a normal NPLF at infinity if there is a density operator ρ such that

$$\omega^\infty(A) = \lim \mathrm{Tr}(\rho A_t) \qquad \forall A \in \mathscr{A}.$$

Theorem 1. Let ρ be a density operator; then the formula

$$\omega_\rho^\infty(A) = \lim \mathrm{Tr}(\rho A_t)$$

defines an NPLF ω_ρ^∞ on $\mathscr{A}$. Also

$$\omega_\rho^\infty(A) = 0 \qquad \forall A \in \mathscr{A}_0^s.$$

Proof. Let $A \in \mathscr{A}$; then $A = T + G$ for some $T \in \mathscr{A}_0^s$, $G \in L^\infty(\boldsymbol{p})$. Then for every $\phi \in \mathscr{H}$ we have $\lim\langle\phi \mid A_t\phi\rangle = \langle\phi \mid G\phi\rangle$.

Let $\{\psi_k\colon k = 1, 2, \ldots\}$ be an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of ρ and for each k let λ_k be the eigenvalue associated with ψ_k.

Since $\Sigma_{k=1}^\infty \lambda_k = 1$ it follows that $\Sigma_{k=1}^\infty \lambda_k\langle\psi_k \mid G\psi_k\rangle$ exists and also that

$$\lim_{t\to\infty} \sum_{k=1}^\infty \lambda_k\langle\psi_k \mid T_t\psi_k\rangle = \sum_{k=1}^\infty \lambda_k\left(\lim_{t\to\infty}\langle\psi_k \mid T_t\psi_k\rangle\right) = 0$$

since the sum is uniform by the Weierstrass M-test (Apostol 1974). Hence

$$\omega_\rho^\infty(A) = \lim_{t\to\infty} \sum_{k=1}^\infty \langle\psi_k \mid \rho A_t\psi_k\rangle$$

$$= \lim_{t\to\infty} \sum_{k=1}^\infty \lambda_k\langle\psi_k \mid (T_t + G)\psi_k\rangle = \sum_{k=1}^\infty \lambda_k\langle\psi_k \mid G\psi_k\rangle$$

so ω_ρ^∞ is well defined.

Clearly ω_ρ^∞ is linear and $\omega_\rho^\infty(I) = \Sigma_{k=1}^\infty \lambda_k = 1$.

To show ω_ρ^∞ is positive observe that if $A = T + G$ as above then

$$\|(T^\dagger G)_t\psi\| = \|T_t^\dagger(G\psi)\|, \qquad \|(GT^\dagger)_t\psi\| \leq \|G\|\|T_t^\dagger\psi\|$$

imply that $T^\dagger T + T^\dagger G + GT^\dagger \in \mathscr{A}_0^s$ so

$$\begin{aligned}\omega_\rho^\infty(A^\dagger A) &= \omega_\rho^\infty(T^\dagger T + T^\dagger G + G^\dagger T + G^\dagger G)\\ &= \omega_\rho^\infty(G^\dagger G)\\ &= \sum_{k=1}^\infty \lambda_k\langle\psi_k \mid G^\dagger G\psi_k\rangle \geq 0.\end{aligned}$$

Finally if $A \in \mathscr{A}_0^s$ then $G = 0$ so $\omega_\rho^\infty(A) = 0$.

Corollary 1. Every normal NPLF ω_ρ generates a normal NPLF at infinity to be denoted by ω_ρ^∞ and every normal NPLF at infinity arises from a normal NPLF.

We shall assume that the states of our system are generated by the normal NPLFS and the normal NPLFS at infinity. So we now replace postulate 2 by the following postulate.

Postulate 3. Any state of the system is represented by a convex combination $\lambda\omega_p + \mu\omega_\sigma^\infty$ ($\lambda, \mu \geqslant 0$; $\lambda + \mu = 1$) of a normal NPLF and a normal NPLF at infinity. Conversely every such convex combination represents a possible state.

Henceforth we shall refer to normal NPLFS as normal states and normal NPLFS at infinity will be called simply states at infinity. If Λ is a bounded Borel set of $\mathbb{R}^n$ and E_x is the spectral measure of position then $E_x(\Lambda) \in \mathscr{A}_0^s$ so for any state at infinity ω_ρ^∞ the expectation value $\omega_\rho^\infty(E_x(\Lambda))$ vanishes, leading to a zero probability of finding the particle in Λ.

Before proceeding further let us carry out the customary classification of states into pure and mixed states.

Definition 3. Pure and mixed states. A state ω is pure if it cannot be decomposed into a non-trivial convex combination of two different states i.e. if

$$0<\lambda<1, \qquad \omega = \lambda\omega_1 + (1-\lambda)\omega_2 \Rightarrow \omega = \omega_1 = \omega_2.$$

Otherwise the state is called mixed and is said to be a mixture of the states ω_1 and ω_2.

Theorem 2. For every unit vector $\phi \in L^2(\mathbb{R}^n)$, ω_ϕ is a pure state on $\mathscr{A}$.

Proof. We know from III that the von Neumann algebra generated by $\mathscr{A}$ is $B(\mathscr{H})$ and it follows that $\mathscr{A}$ is irreducible and hence ϕ is a cyclic vector for $\mathscr{A}$.

Let $i: \mathscr{A} \to B(\mathscr{H})$ be the inclusion map $i(A) = A$; then i is clearly an irreducible cyclic representation of $\mathscr{A}$ with cyclic vector ϕ such that $\omega_\phi(A) = \langle\phi | i(A)\phi\rangle$.

Hence i is unitarily equivalent to the GNS representation associated with ω_ϕ and so the latter is also irreducible. It follows that ω_ϕ is pure.

The steps in this argument are justified by proposition 2.3.8 and theorems 2.3.16 and 2.3.19 of Bratteli and Robinson (1979).

Definition 4. Asymptotically disjoint states. Two normal states ω_{ρ_1} and ω_{ρ_2} are said to be asymptotically disjoint if $\lim\langle\phi_t | A\psi_t\rangle = 0$ for all $A \in \mathscr{A}$ whenever ϕ is an eigenvector of ρ_1 and ψ is an eigenvector of ρ_2. The corresponding states at infinity, $\omega_{\rho_1}^\infty$ and $\omega_{\rho_2}^\infty$, are said to be disjoint.

In particular ω_ϕ and ω_ψ are asymptotically disjoint if $\lim\langle\phi_t | A\psi_t\rangle = 0$ for all $A \in \mathscr{A}$, and ω_ϕ^∞ and ω_ψ^∞ are disjoint.

Before presenting the central theorems of our formulation we prove a lemma.

Lemma 1. Let $\phi, \psi \in L^2(\mathbb{R}^n)$ and let E_x be the spectral measure of position; then these are equivalent:

(1) $\forall$ Borel sets of $\mathbb{R}^n$, $\langle\phi | E_x(\Lambda)\psi\rangle = 0$.

(2) There are disjoint Borel sets Λ_1, Λ_2 of $\mathbb{R}^n$ with $E_x(\Lambda_1)\phi = \phi$, $E_x(\Lambda_2)\psi = \psi$.

Proof. We need only show [1]$\Rightarrow$[2] the converse being obvious. So assume $\int_\Lambda \bar{\phi}\psi = 0$ for all Λ.

Let u be the real part of $\bar{\phi}\psi$ and let u^+, u^- be the positive and negative parts of u; then

$$\int_\Lambda u = 0 \qquad \forall\Lambda.$$

Let $\Lambda^+ = \{x: u^+(x) \neq 0\}$ and $\Lambda^- = \{x: u^-(x) \neq 0\}$; then Λ^+, Λ^- are measurable and $\Lambda^+ \cap \Lambda^-$ is empty. Hence

$$\int u^+ = \int_{\Lambda^+} u^+ = \int_{\Lambda^+} u = 0,$$

so $u^+ = 0$ almost everywhere. Similarly $u^- = 0$ almost everywhere and an identical argument applied to the imaginary part of $\bar{\phi}\psi$ gives $\bar{\phi}\psi = 0$ almost everywhere.

Thus $\phi(x) = 0$ or $\psi(x) = 0$ for almost all $x \in \mathbb{R}^n$. Now let $\Lambda_1 = \{x: \phi(x) \neq 0\}$, $\Lambda_2 = \{x: \psi(x) \neq 0\}$; then Λ_1, Λ_2 are measurable, $E_x(\Lambda_1)\phi = \phi$, $E_x(\Lambda_2)\psi = \psi$ and

$$\Lambda_1 \cap \Lambda_2 = \{x: \phi(x) \neq 0 \quad \text{and} \quad \psi(x) \neq 0\}$$

which has measure zero. (2) follows from this.

Theorem 3. The pure states ω_ϕ and ω_ψ on $\mathscr{A}$ are asymptotically disjoint if and only if they correspond to disjoint momentum values i.e. if and only if there are disjoint Borel sets Λ_1 and Λ_2 of $\mathbb{R}^n$ with $E_p(\Lambda_1)\phi = \phi$ and $E_p(\Lambda_2)\psi = \psi$.

Proof.

(i) Suppose ϕ and ψ correspond to disjoint momentum values; then for all $B \in L^\infty(p)$

$$\lim\langle\phi \mid B_t\psi\rangle = \langle\phi \mid B\psi\rangle = 0.$$

Also for all $A \in \mathscr{A}_0^s$ we have $\lim\|A\psi_t\| = 0$ so $\lim\langle\phi \mid A_t\psi\rangle = 0$. Since $\mathscr{A} = \mathscr{A}_0^s + L^\infty(p)$ it follows that ω_ϕ, ω_ψ are asymptotically disjoint.

(ii) If ω_ϕ and ω_ψ are asymptotically disjoint then for any spectral projector $E_p(\Lambda)$ of momentum

$$\langle\phi \mid E_p(\Lambda)\psi\rangle = \lim\langle\phi \mid (E_p(\Lambda))_t\psi\rangle = 0$$

and it follows from lemma 1 and the relation $E_p(\hbar^{-1}\Lambda) = F^\dagger E_x(\Lambda)F$ (Wan and McLean 1984) that ϕ and ψ correspond to disjoint momentum values.

We recall from papers I and II that $\phi, \psi \in L^2(\mathbb{R}^n)$ are said to be asymptotically separable if there are disjoint n-dimensional cuboids $[v, w]$ and $[v', w']$ in $\mathbb{R}^n$ such that ϕ, ψ are asymptotically localisable in the disjoint regions $[vt, wt]$ and $[v't, w't]$ respectively, i.e.

$$\lim\|E_x[vt, wt]\phi_t\|^2 = 1 = \lim\|E_x[v't, w't]\psi_t\|^2.$$

Corollary 2.

(1) *If ϕ, ψ* are asymptotically separable then ω_ϕ and ω_ψ are asymptotically disjoint.

(2) Two states ω_{ρ_1} and ω_{ρ_2} are asymptotically disjoint iff ϕ and ψ correspond to disjoint momentum values whenever ϕ is an eigenvector of ρ_1 and ψ is an eigenvector of ρ_2.

Proof.

(1) follows from corollary in II while (2) is obvious.

Theorem 4. Let ϕ, ϕ_1 and ϕ_2 be unit vectors in $L^2(\mathbb{R}^n)$ such that ϕ_1 and ϕ_2 correspond to disjoint momentum values and $\phi = \lambda\phi_1 + \mu\phi_2$ where $\lambda, \mu \neq 0$. Then $\omega_\phi^\infty = |\lambda|^2\omega_{\phi_1}^\infty + |\mu|^2\omega_{\phi_2}^\infty$.

Proof. By theorem 3 we have

$$\lim\langle\phi_1|A_t\phi_2\rangle = 0 = \lim\langle\phi_2|A_t\phi_1\rangle \qquad \forall A\in\mathscr{A},$$

so

$$\begin{aligned}\omega_\phi^\infty(A) &= \lim\langle\phi|A_t\phi\rangle\\ &= \lim(|\lambda|^2\langle\phi_1|A_t\phi_1\rangle + |\mu|^2\langle\phi_2|A_t\phi_2\rangle)\\ &= |\lambda|^2\omega_{\phi_1}^\infty(A) + |\mu|^2\omega_{\phi_2}^\infty(A).\end{aligned}$$

Theorem 5.

(1) Every state at infinity on $\mathscr{A}$ is mixed.

(2) A state on $\mathscr{A}$ is pure if and only if it is of the form ω_ϕ for some $\phi\in L^2(\mathbb{R}^n)$.

(3) Every pure state evolves into a mixed state as $t\to\infty$.

Proof.

(1) Follows from theorem 4.

(2) Each ω_ϕ is pure by theorem 2.

By postulate 3 every state is a convex combination of a normal state and a state at infinity. Since no state at infinity is equal to a normal state it follows from (1) that any pure state must be normal. If ρ is a density operator which is not a one-dimensional projector it is easily verified that ω_ρ is mixed and hence every pure state is of the form ω_ϕ.

(3) Clearly ω_ϕ evolves into ω_ϕ^∞ and now the assertion follows from (1) and (2).

We conclude this section on states by noting that any two states of the form ω_ϕ, ω_ψ are coherent in the sense that $\langle\phi|A\psi\rangle$ is non-zero for some $A\in\mathscr{A}$. For if A is the compact operator defined by

$$Af = \langle\psi|f\rangle\phi \qquad \forall f\in\mathscr{H}$$

then $A\in\mathscr{A}$ by theorem 5 of III and clearly $\langle\phi|A\psi\rangle\neq 0$.

Finally we would point out the easily verified results that $\mathscr{A}_0^s$ is a subalgebra of $\mathscr{A}$ as well as a closed two-sided ideal in $\mathscr{A}$, and that the quotient algebra $\mathscr{A}/\mathscr{A}_0^s$ is isomorphic to $L^\infty(\boldsymbol{p})$ and hence to $L^\infty(\mathbb{R}^n)$. Positive normalised functions in $L^1(\mathbb{R}^n)$ correspond to normal NPLFS at infinity.

3.4. *Time evolution*

Following the usual practice we make the following postulate.

Postulate 4. The time evolution of the system is described by a one-parameter group $\{\alpha_t\colon t\in\mathbb{R}\}$ of automorphisms of $\mathscr{A}$.

For the system under consideration, i.e. a free quantum particle, we have

$$\alpha_t(A) = U_t^{0+}AU_t^0 \qquad \forall A\in\mathscr{A}$$

(α_t is an automorphism since $U_t^0\in\mathscr{A}$).

In the Heisenberg picture the time evolution of an observable A is given by $A_t = \alpha_t(A)$. We can also work in the Schrödinger picture in which a state ω evolves in time according to $\omega_t(A) = \omega(\alpha_t(A))$.

It follows that $(\omega_\phi)_t = \omega_{\phi_t}$ so that we recover the usual quantum mechanical evolution of states. Two special cases present themselves:

(1) in the Heisenberg picture $A \in L^\infty(\boldsymbol{p})$ does not evolve in time,

(2) in the Schrödinger picture states at infinity do not evolve in time.

Case (1) is obvious. Since an arbitrary element of $\mathscr{A}$ is of the form $A = T + G$ with $T \in \mathscr{A}_0^s$ and $G \in L^\infty(\boldsymbol{p})$ we have $\forall t \in \mathbb{R}$

$$\begin{aligned}\omega_t^\infty(A) &= \omega^\infty(\alpha_t(T)) + \omega^\infty(\alpha_t(G))\\ &= \omega^\infty(G)\\ &= \omega^\infty(A)\end{aligned}$$

verifying assertion (2).

Finally we shall say that a state ω evolves in time into a state at infinity ω^∞ if $\lim \omega_t(A) = \omega^\infty(A)\ \forall A \in \mathscr{A}$.

4. Quantum separability and the de Broglie paradox

Let us examine the physical implications of postulates 1 and 3 and of the results subsequently obtained in the preceding section. By restricting the C^*-algebra associated with the system to $\mathscr{A}$ and the introduction of states at infinity the theory possesses a feature which is absent in the conventional quantum mechanics, namely, the theory is asymptotically separable in the sense that:

(i) A pure state can evolve in time into a mixture.

(ii) Two coherent pure states can evolve in time into two disjoint states.

(iii) The reason for (i), (ii) above is due to asymptotically vanishing correlations between spatially infinitely separating states.

The separability of the theory enables us to tackle those age old problems not readily soluble in the conventional non-separable quantum theory. Let us briefly list some of these problems.

(1) The transition of a pure state into a mixture (arising from the von Neumann projection postulate) after a measurement as exemplified by the paradox of Schrödinger's cat (Wan 1980). Conventional quantum mechanics forbids such a transition but our theory explicitly allows such a transition, albeit asymptotically.

(2) In conventional quantum mechanics a coherent superposition of two states $\lambda\phi + \mu\psi$ is pure for all times while our formulation allows such a coherent superposition of certain states to evolve into a mixture. The de Broglie paradox (Selleri and Tarozzi 1981) ceases to be a problem in the present theory.

(3) The characterisation of a system at infinity in our theory becomes very simple. At large times and consequently at large distances away from the origin, a quantum system is characterisable, say to an observer near the origin, in the present theory in the following fashion:

(i) States: The system at infinity admits only states at infinity. These states give zero expectation value with respect to quasi-local observables.

(ii) Observables: Only observables in $L^\infty(\boldsymbol{p})$ can give non-zero expectation values with respect to ω^∞. Hence only observables in $L^\infty(\boldsymbol{p})$ are relevant to a system at infinity. We call observables in $L^\infty(\boldsymbol{p})$, and also $\boldsymbol{p}$ and H_0 observables at infinity.

(iii) Asymptotic superselection rule: A notion closely related to observables at infinity is that of superselectors at infinity (Wan 1980) introduced below.

Definition 5. Superselectors at infinity. A bounded operator $Q^\infty \neq 0$ on $\mathscr{H}$ is called a superselector at infinity (or simply superselector for short) if

(1) $Q^\infty = s\text{-lim}\, Q_t$ for some $Q \in B(\mathscr{H})$, and

(2) $s\text{-lim}[Q, A]_t = 0\ \forall A \in \mathscr{A}$.

Note that (2) above is equivalent to the requirement that $s\text{-lim}[Q_t, A_t] = 0$ for all A in $\mathscr{A}$.

Theorem 6. A bounded non-zero operator Q^∞ on $\mathscr{H}$ is a superselector at infinity if and only if $Q^\infty \in L^\infty(\boldsymbol{p})$.

Proof. Let $A^\infty = s\text{-lim}\, A_t$. Then $s\text{-lim}[Q_t, A_t] = [Q^\infty, A^\infty] = 0$. Since $A^\infty \in L^\infty(\boldsymbol{p})$ by theorem 8 of III we have Q^∞ lying in the commutant of the von Neumann algebra $L^\infty(\boldsymbol{p})$. Hence $Q^\infty \in L^\infty(\boldsymbol{p})$ and similarly for the sufficient condition.

We see that self-adjoint superselectors at infinity coincide with observables at infinity. We can also introduce unbounded self-adjoint superselectors by the requirement that their spectral projectors be superselectors. Theorem 6 tells us that there is really only one independent (unbounded) superselector which is the momentum $\boldsymbol{p}$. Following common practice we can also introduce supersectors (Wan 1980).

Definition 6. Supersectors at infinity. A supersector at infinity, S_Λ, of the superselector at infinity, $\boldsymbol{p}$, is the set of states at infinity, ω_ϕ^∞, arising from all the normal states ϕ lying in the subspace of $\mathscr{H}$ associated with the spectral projector $E_{\boldsymbol{p}}(\Lambda)$.

Obviously two supersectors S_{Λ_1}, S_{Λ_2} are disjoint if Λ_1, Λ_2 are disjoint.

To conclude this section we point out that terms like observables at infinity, classical observables, macroscopic observables, superselectors have been introduced by various authors (Lanford and Ruelle 1969, Hepp 1972, Primas and Müller-Herold 1978, Wan 1980) with differing meanings.

5. Concluding remarks and prospects

We should point out that the formulation presented so far is not meant to be a rigid and final structure. In fact the formulation is flexible and it can be extended or amended easily. The following suffices to illustrate this. Firstly we can extend the present formulation for a free particle to a simple scattering system. This can be achieved in a similar fashion to I and II, namely by restricting states to scattering states only. Secondly the present theory for a free particle itself may be extended, for example, by enlarging the set of states. One class of additional states comes readily to mind. Let $C_0(\boldsymbol{p})$ be the set of operators of the form $f(\boldsymbol{p})$ where f is continuous and vanishes at infinity. Then $C_0(\boldsymbol{p}) \subset L^\infty(\boldsymbol{p})$ and for each $\boldsymbol{k} \in \mathbb{R}^n$ we can define an NPLF $\Omega_{\boldsymbol{k}}^\infty$ on $C_0(\boldsymbol{p})$ by

$$\Omega_{\boldsymbol{k}}^\infty(f(\boldsymbol{p})) = f(\hbar \boldsymbol{k}),$$

then $\Omega_{\boldsymbol{k}}^\infty$ is pure and may be extended to a pure NPLF on $L^\infty(\boldsymbol{p})$ (Segal 1947). $\Omega_{\boldsymbol{k}}^\infty$ may then be extended to $\mathscr{A}$, since every $A \in \mathscr{A}$ is of the form $T + G$ with $T \in \mathscr{A}_0^s$, $G \in L^\infty(\boldsymbol{p})$, by defining

$$\Omega_{\boldsymbol{k}}^\infty(T + G) = \Omega_{\boldsymbol{k}}^\infty(G)$$

and it is easily verified that $\Omega_{\boldsymbol{k}}^\infty$ is a pure state at infinity.

Finally we can extend the present theory to a many particle system. The asymptotic separability of the resulting theory would enable us to tackle problems like the EPR paradox (Einstein *et al* 1935), and the quantum measurement problem. Work along this line is continuing.

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